The centralizer of an element in an endomorphism ring

Jenő Szigeti and Leon van Wyk

Abstract. We prove that the centralizer $C_\varphi \subseteq \mathrm{Hom}_R(M,M)$ of a nilpotent endomorphism φ of a finitely generated semisimple left R-module $_RM$ (over an arbitrary ring R) is the homomorphic image of the opposite of a certain $Z(R)$-subalgebra of the full $m\times m$ matrix algebra $M_{m\times m}(R[t])$, where m is the dimension (composition length) of $\ker(\varphi)$. If R is a finite dimensional division ring over its central subfield $Z(R)$ and φ is nilpotent, then we give an upper bound for the $Z(R)$-dimension of C_φ. If R is a local ring, φ is nilpotent and $\sigma \in\mathrm{Hom}_R(M,M)$ is arbitrary, then we provide a complete description of the containment $C_\varphi \subseteq C_\sigma$ in terms of an appropriate R-generating set of $_RM$. For an arbitrary (not necessarily nilpotent) linear map $\varphi \in\mathrm{Hom}_K(V,V)$ of a finite dimensional vector space V over an algebraically closed field K we prove that C_φ is the homomorphic image of a direct product of p factors such that for each $1\leq i\leq p$ the i-th factor is a K-subalgebra of $M_{m_i\times m_i}(K[t])$ with $m_i=\dim(\ker(\varphi-\lambda_i 1_V))$ and $\{\lambda_1,\lambda_2,\dots,\lambda_p\}$ is the set of all eigenvalues of φ. As a consequence, we obtain that C_φ satisfies all polynomial identities of $M_{m\times m}(K[t])$, where m is the maximum of the m_i's.

1. INTRODUCTION

Our work was motivated by one of the classical subjects of advanced linear algebra. A detailed study of commuting matrices can be found in many of the text books on linear algebra ([5,6]). Commuting pairs and k-tuples of $n\times n$ matrices have been continuously in the focus of research (see, for example, [2,3,4]). If we replace $n\times n$ matrices by endomorphisms of an n-generated module, we get a more general situation. The aim of the present paper is to investigate the size and the PI properties of the centralizer C_φ of an element φ in the endomorphism ring $\mathrm{Hom}_R(M,M)$.

In the case of a finitely generated semisimple left R-module $_RM$ (over an arbitrary ring R) we obtain results about the centralizer of a nilpotent endomorphism φ. First we prove that C_φ is the homomorphic image of the opposite of a certain

1991 *Mathematics Subject Classification.* 15A27, 16D60, 16S50, 16U70.

Key words and phrases. centralizer, module endomorphism, induced module structure, nilpotent Jordan normal base,

The first author was supported by OTKA of Hungary No. K61007. The second author was supported by the National Research Foundation of South Africa under Grant No. UID 61857. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the National Research Foundation does not accept any liability in regard thereto.

$Z(R)$-subalgebra of the full $m \times m$ matrix algebra $M_{m\times m}(R[t])$ over the polynomial ring $R[t]$, where m is the dimension (composition length) of $\ker(\varphi)$. As a consequence, we obtain that C_φ satisfies all polynomial identities of $M^{\mathrm{op}}_{m\times m}(R[t])$; in particular, if R is commutative, then the standard identity $S_{2m} = 0$ of degree $2m$ holds in C_φ. The authors do not know similar results in the literature.

Then we exhibit the centralizer C_φ of a nilpotent φ as a homomorphic image of the intersection of some md-generated R-submodule of $M_{m\times m}(R[t])$ and of the above mentioned $Z(R)$-subalgebra, where d is the dimension (composition length) of ${}_RM$. If R is a finite dimensional division ring over its central subfield $Z(R)$, then we can use this homomorphic image representation to derive a "sharp" upper bound for the $Z(R)$-dimension of C_φ. For an arbitrary (not necessarily nilpotent) linear map of a finite dimensional vector space, or equivalently for a matrix $A \in M_{n\times n}(K)$ over an algebraically closed field K, the dimension of C_A is well known (see [5,6]). A multiplicative (K vector space) base of C_A was constructed in [4].

If $\varphi : M \longrightarrow M$ is a so called indecomposable nilpotent R-endomorphism, then we give a complete description of C_φ in terms of an appropriate R-generating set of ${}_RM$. In particular, if R is commutative, then we prove that $\psi \in C_\varphi$ holds if and only if ψ is a polynomial of φ. A nilpotent linear map (over an algebraically closed field) is indecomposable if and only if its characteristic polynomial coincides with the minimal polynomial. The following is a classical result about the centralizer (see [5] part VII, section 39). If K is an algebraically closed field and the characteristic polynomial of a (not necessarily nilpotent) $A \in M_{n\times n}(K)$ coincides with the minimal polynomial, then

$$C_A = \{f(A) \mid f(t) \in K[t]\}.$$

For an indecomposable nilpotent φ our description of C_φ is a generalization of the above result. We note that the above result on C_A is similar to Bergman's Theorem ([1]) about the centralizer of a non-constant polynomial in the free associative algebra.

The containment relation $C_\varphi \subseteq C_\sigma$ is also considered. If R is a local ring, φ is nilpotent and $\sigma \in \mathrm{Hom}_R(M, M)$ is arbitrary, then we provide a complete description of the situation $C_\varphi \subseteq C_\sigma$ in terms of an appropriate R-generating set of ${}_RM$. For two (not necessarily nilpotent) matrices $A, B \in M_{n\times n}(K)$, over an algebraically closed field K, the containment $C_A \subseteq C_B$ holds if and only if $B = f(A)$ for some $f(t) \in K[t]$ (see [5] part VII, section 39). For a nilpotent φ our description of $C_\varphi \subseteq C_\sigma$ is a generalization of the above result.

In the case of a finite dimensional vector space V over an algebraically closed field K we obtain results about the centralizer of an arbitrary (not necessarily nilpotent) linear map $\varphi \in \mathrm{Hom}_K(V, V)$. We prove that C_φ is the homomorphic image of a direct product of p factors such that for each $1 \le i \le p$ the i-th factor is a K-subalgebra of $M_{m_i\times m_i}(K[t])$, where

$$m_i = \dim(\ker(\varphi - \lambda_i 1_V))$$

and $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is the set of all eigenvalues of φ. As a consequence, we obtain that C_φ satisfies all polynomial identities of $M_{m\times m}(K[t])$, where m is the maximum of the m_i's. The following related results can be found in [4,6]. The centralizer K-algebra C_A of a matrix $A \in M_{n\times n}(K)$ (over an algebraically closed field K) is isomorphic to the direct product of the centralizers C_{A_i}, where A_i denotes the block diagonal matrix consisting of all Jordan blocks of A having eigenvalue λ_i in

the diagonal. The number of the diagonal blocks in A_i is $m_i = \dim(\ker(A - \lambda_i I))$ and the size of A_i is $d_i \times d_i$, where d_i is the multiplicity of the root λ_i in the characteristic polynomial of A. For a matrix $X \in M_{d_i \times d_i}(K)$ we have $X \in C_{A_i}$ if and only if $X = [X_{k,l}]$ is an $m_i \times m_i$ matrix of blocks, where $X_{k,l}$ is an arbitrary triangularly striped matrix block of size $s_i(k) \times s_i(l)$ and $s_i(k)$ denotes the size of the k-th elementary Jordan block in A_i for $1 \leq k \leq m_i$. We note that $\dim(C_{A_i}) = s_i(1) + 3s_i(2) + \cdots + (2m_i - 1)s_i(m_i)$ provided that $s_i(1) \geq s_i(2) \geq \cdots \geq s_i(m_i)$. It seems that we cannot use the striped block structure of the elements of C_{A_i} to derive the above mentioned PI properties of C_A ($\cong C_\varphi$), only the much weaker statement that C_φ (or C_A) satisfies all polynomial identities of $M_{d \times d}(K)$ follows (d is the maximum of the d_i's and $d_i = s_i(1) + s_i(2) + \cdots + s_i(m_i)$). The above consideration confirms that the use of the polynomial ring $K[t]$ is an essential ingredient of our treatment.

Since all known results about centralizers are in close connection with the Jordan normal base, it is not surprising that our development depends on the existence of the (nilpotent) Jordan normal base of a semisimple module with respect to a given nilpotent endomorphism (guaranteed by one of the main theorems of [7]). Using an endomorphism of ${}_RM$, a natural $R[t]$-module structure on M can be defined. Almost all of our proofs are based on calculations using this induced module structure. In the proof of the last theorem we use the Fitting Lemma.

2. THE NILPOTENT JORDAN NORMAL BASE

Throughout the paper a ring R means a (not necessarily commutative) ring with identity, $Z(R)$ and $J(R)$ denote the centre and the Jacobson radical of R, respectively. Let $\varphi : M \longrightarrow M$ be an R-endomorphism of the (unitary) left R-module ${}_RM$. A subset

$$\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\} \subseteq M$$

is called a *nilpotent Jordan normal base* of ${}_RM$ with respect to φ if each R-submodule $Rx_{\gamma,i} \leq M$ is simple,

$$\bigoplus_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} Rx_{\gamma,i} = M$$

is a direct sum, $\varphi(x_{\gamma,i}) = x_{\gamma,i+1}$, $\varphi(x_{\gamma,k_\gamma}) = x_{\gamma,k_\gamma+1} = 0$ for all $\gamma \in \Gamma$, $1 \leq i \leq k_\gamma$, and the set $\{k_\gamma \mid \gamma \in \Gamma\}$ of integers is bounded. For $i \geq k_\gamma + 1$ we assume that $x_{\gamma,i} = 0$ holds in M. Now Γ is called the set of (Jordan-) blocks and the size of the block $\gamma \in \Gamma$ is the integer $k_\gamma \geq 1$. Obviously, the existence of a nilpotent Jordan normal base implies that ${}_RM$ is semisimple and φ is nilpotent with $\varphi^n = 0 \neq \varphi^{n-1}$, where

$$n = \max\{k_\gamma \mid \gamma \in \Gamma\}$$

is the index of nilpotency. If ${}_RM$ is finitely generated, then

$$\sum_{\gamma \in \Gamma} k_\gamma = \dim_R(M)$$

is the dimension of ${}_RM$ (equivalently: the composition length of ${}_RM$ or the height of the submodule lattice of ${}_RM$). Clearly,

$$\varphi(M) = \varphi\left(\sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} Rx_{\gamma,i}\right) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma} R\varphi(x_{\gamma,i}) = \sum_{\gamma \in \Gamma, 1 \leq i \leq k_\gamma - 1} Rx_{\gamma,i+1}$$

implies that

$$\operatorname{im}(\varphi) = \underset{\gamma\in\Gamma,1\leq i\leq k_\gamma-1}{\oplus} Rx_{\gamma,i+1} = \underset{\gamma\in\Gamma',2\leq i'\leq k_\gamma}{\oplus} Rx_{\gamma,i'},$$

where

$$\Gamma' = \{\gamma \in \Gamma \mid k_\gamma \geq 2\} \text{ and } \Gamma \setminus \Gamma' = \{\gamma \in \Gamma \mid k_\gamma = 1\}.$$

Any element $u \in M$ can be written as

$$u = \sum_{\gamma\in\Gamma,1\leq i\leq k_\gamma} a_{\gamma,i}x_{\gamma,i},$$

where $\{(\gamma, i) \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma, \text{ and } a_{\gamma,i} \neq 0\}$ is finite and all summands $a_{\gamma,i}x_{\gamma,i}$ are uniquely determined by u. Since

$$\varphi(u) = \sum_{\gamma\in\Gamma,1\leq i\leq k_\gamma} a_{\gamma,i}\varphi(x_{\gamma,i}) = \sum_{\gamma\in\Gamma,1\leq i\leq k_\gamma} a_{\gamma,i}x_{\gamma,i+1} = 0$$

is equivalent to the condition that $a_{\gamma,i}x_{\gamma,i+1} = 0$ for all $\gamma \in \Gamma, 1 \leq i \leq k_\gamma - 1$, we obtain that

$$\varphi(u) = 0 \Longleftrightarrow u = \sum_{\gamma\in\Gamma} a_{\gamma,k_\gamma}x_{\gamma,k_\gamma}.$$

Indeed, $a_{\gamma,i}x_{\gamma,i} \neq 0$ $(1 \leq i \leq k_\gamma - 1)$ would imply that $ba_{\gamma,i}x_{\gamma,i} = x_{\gamma,i}$ for some $b \in R$ (note that $Rx_{\gamma,i} \leq M$ is simple), whence

$$x_{\gamma,i+1} = \varphi(x_{\gamma,i}) = ba_{\gamma,i}\varphi(x_{\gamma,i}) = ba_{\gamma,i}x_{\gamma,i+1} = 0$$

can be derived, a contradiction. It follows that

$$\ker(\varphi) = \underset{\gamma\in\Gamma}{\oplus} Rx_{\gamma,k_\gamma}$$

and $\dim_R(\ker(\varphi)) = |\Gamma|$ in case of a finite Γ. The following is one of the main results in [7].

2.1.Theorem. *Let $\varphi : M \longrightarrow M$ be an R-endomorphism of the left R-module ${}_RM$. Then the following are equivalent.*

(1) *${}_RM$ is a semisimple left R-module and φ is nilpotent.*
(2) *There exists a nilpotent Jordan normal base of ${}_RM$ with respect to φ.*

2.2.Proposition. *Let $\varphi : M \longrightarrow M$ be a nilpotent R-endomorphism of the finitely generated semisimple left R-module ${}_RM$. If $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ and $\{y_{\delta,j} \mid \delta \in \Delta, 1 \leq j \leq l_\delta\}$ are nilpotent Jordan normal bases of ${}_RM$ with respect to φ, then there exists a bijection $\pi : \Gamma \longrightarrow \Delta$ such that $k_\gamma = l_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. Thus the sizes of the blocks of a nilpotent Jordan normal base are unique up to a permutation of the blocks.*

Proof. We apply induction on the index of the nilpotency of φ. If $\varphi = 0$, then we have $k_\gamma = l_\delta = 1$ for all $\gamma \in \Gamma$, $\delta \in \Delta$, and

$$\underset{\gamma\in\Gamma}{\oplus} Rx_{\gamma,1} = \underset{\delta\in\Delta}{\oplus} Ry_{\delta,1} = M$$

implies the existence of a bijection $\pi : \Gamma \longrightarrow \Delta$ (Krull-Schmidt, Kuros-Ore). Assume that our statement holds for any R-endomorphism $\phi : N \longrightarrow N$ with ${}_RN$

being a finitely generated semisimple left R-module and $\phi^{n-2} \neq 0 = \phi^{n-1}$. Consider the situation described in the proposition with $\varphi^{n-1} \neq 0 = \varphi^n$, then

$$\mathrm{im}(\varphi) = \bigoplus_{\gamma \in \Gamma', 2 \leq i' \leq k_\gamma} Rx_{\gamma,i'}$$

ensures that

$$\{x_{\gamma,i'} \mid \gamma \in \Gamma', 2 \leq i' \leq k_\gamma\}$$

is a nilpotent Jordan normal base of the left R-submodule $\mathrm{im}(\varphi) \leq M$ of ${}_RM$ with respect to the restricted R-endomorphism $\varphi : \mathrm{im}(\varphi) \longrightarrow \mathrm{im}(\varphi)$. The same holds for

$$\{y_{\delta,j'} \mid \delta \in \Delta', 2 \leq j' \leq l_\delta\}.$$

Since we have $\phi^{n-2} \neq 0 = \phi^{n-1}$ for $\phi = \varphi \restriction \mathrm{im}(\varphi)$, our assumption ensures the existence of a bijection $\pi : \Gamma' \longrightarrow \Delta'$ such that $k_\gamma - 1 = l_{\pi(\gamma)} - 1$ for all $\gamma \in \Gamma'$. In view of

$$\ker(\varphi) = \bigoplus_{\gamma \in \Gamma} Rx_{\gamma,k_\gamma} = \bigoplus_{\delta \in \Delta} Ry_{\delta,l_\delta}$$

we obtain that $|\Gamma| = |\Delta|$ (Krull-Schmidt, Kuros-Ore), whence $|\Gamma \setminus \Gamma'| = |\Delta \setminus \Delta'|$ follows. Thus we have a bijection $\pi^* : \Gamma \setminus \Gamma' \longrightarrow \Delta \setminus \Delta'$ and the natural map

$$\pi \sqcup \pi^* : \Gamma' \cup (\Gamma \setminus \Gamma') \longrightarrow \Delta' \cup (\Delta \setminus \Delta')$$

is a bijection with the desired property.□

We call a nilpotent element $s \in S$ of the ring S *decomposable* if $es = se$ holds for some idempotent element $e \in S$ ($e^2 = e$) with $0 \neq e \neq 1$. A nilpotent element which is not decomposable is called *indecomposable*.

2.3.Proposition. *Let $\varphi : M \longrightarrow M$ be a nonzero nilpotent R-endomorphism of the semisimple left R-module ${}_RM$. Then the following are equivalent.*

(1) *There is a nilpotent Jordan normal base $\{x_i \mid 1 \leq i \leq n\}$ of ${}_RM$ with respect to φ consisting of one block (thus $|\Gamma| = 1$ for any nilpotent Jordan normal base $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ of ${}_RM$ with respect to φ).*
(2) *φ is an indecomposable nilpotent element of the ring $\mathrm{Hom}_R(M,M)$.*
(3) *${}_RM$ is finitely generated and $\varphi^{d-1} \neq 0$, where $d = \dim_R(M)$ is the dimension of ${}_RM$.*

Proof.
(1)$\Longrightarrow$(3): Clearly,

$$\bigoplus_{1 \leq i \leq n} Rx_i = M$$

implies that we have $d = n$ for the dimension of ${}_RM$, whence

$$\varphi^{d-1}(x_1) = \varphi^{n-1}(x_1) = x_n \neq 0$$

follows.
(3)$\Longrightarrow$(1): Let $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ be a Jordan normal base of ${}_RM$ with respect to φ. Suppose that $|\Gamma| \geq 2$, then

$$n = \max\{k_\gamma \mid \gamma \in \Gamma\} \leq d - 1,$$

where $d = \sum_{\gamma \in \Gamma} k_\gamma = \dim_R(M)$. Thus $\varphi^{d-1} = \varphi^{(d-1)-n} \circ \varphi^n = 0$, a contradiction.

(1)$\Longrightarrow$(2): Suppose that $\varepsilon \circ \varphi = \varphi \circ \varepsilon$ holds for some idempotent endomorphism $\varepsilon \in \mathrm{Hom}_R(M, M)$ with $0 \neq \varepsilon \neq 1$. Then

$$\mathrm{im}(\varepsilon) \oplus \mathrm{im}(1 - \varepsilon) = M$$

for the non-zero (semisimple) R-submodules $\mathrm{im}(\varepsilon)$ and $\mathrm{im}(1 - \varepsilon)$ of ${}_R M$. Now $\varepsilon \circ \varphi = \varphi \circ \varepsilon$ ensures that $\varphi : \mathrm{im}(\varepsilon) \longrightarrow \mathrm{im}(\varepsilon)$ and $\varphi : \mathrm{im}(1 - \varepsilon) \longrightarrow \mathrm{im}(1 - \varepsilon)$. Since these restricted R-endomorphisms are nilpotent, we have a nilpotent Jordan normal base of $\mathrm{im}(\varepsilon)$ with respect to $\varphi \restriction \mathrm{im}(\varepsilon)$ and a nilpotent Jordan normal base of $\mathrm{im}(1 - \varepsilon)$ with respect to $\varphi \restriction \mathrm{im}(1 - \varepsilon)$. The union of these two bases gives a nilpotent Jordan normal base of M with respect to φ consisting of more than one block, a contradiction (the direct sum property of the new base is a consequence of the modularity of the submodule lattice of ${}_R M$).
(2)$\Longrightarrow$(1): Suppose that $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ is a nilpotent Jordan normal base of ${}_R M$ with respect to φ with $|\Gamma| \geq 2$ and fix an element $\delta \in \Gamma$. Consider the non-zero φ-invariant R-submodules

$$N'_\delta = \underset{1 \leq i \leq k_\delta}{\oplus} R x_{\delta,i} \text{ and } N''_\delta = \underset{\gamma \in \Gamma \setminus \{\delta\}, 1 \leq i \leq k_\gamma}{\oplus} R x_{\gamma,i} \ ,$$

then $M = N'_\delta \oplus N''_\delta$ and define $\varepsilon_\delta : M \longrightarrow M$ as the natural projection of M onto N'_δ. Then $\varepsilon_\delta(u) = u'$, where $u = u' + u''$ is the unique sum presentation of $u \in M$ with $u' \in N'_\delta$ and $u'' \in N''_\delta$. It is straightforward to see that $\varepsilon_\delta \circ \varepsilon_\delta = \varepsilon_\delta$, $0 \neq \varepsilon_\delta \neq 1$ and $\varepsilon_\delta \circ \varphi = \varphi \circ \varepsilon_\delta$ hold.$\square$

3. THE MODULE STRUCTURE INDUCED BY AN ENDOMORPHISM

Let $R[t]$ denote the ring of polynomials of the commuting indeterminate t with coefficients in R. The ideal $(t^k) = R[t]t^k = t^k R[t] \lhd R[t]$ generated by t^k will be considered in the sequel. If $\varphi : M \longrightarrow M$ is an arbitrary R-endomorphism of the left R-module ${}_R M$, then for $u \in M$ and

$$f(t) = a_1 + a_2 t + \cdots + a_{n+1} t^n \in R[t]$$

(unusual use of indices!) the left multiplication

$$f(t) * u = a_1 u + a_2 \varphi(u) + \cdots + a_{n+1} \varphi^n(u)$$

defines a natural left $R[t]$-module structure on M. This left action of $R[t]$ on M extends the left action of R. The proof of

$$g(t) * (f(t) * u) = (g(t) f(t)) * u$$

is straightforward. Note that

$$t^n * u = \varphi^n(u) \text{ and } \varphi(f(t) * u) = (t f(t)) * u.$$

If $\varphi^k(u) = 0$ for some $1 \leq k \leq n$, then

$$f(t) * u = f^{(k)}(t) * u,$$

where

$$f^{(k)}(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} \in R[t]$$

is the k-cut of $f(t)$. For any R-endomorphism $\psi \in \mathrm{Hom}_R(M, M)$ with $\psi \circ \varphi = \varphi \circ \psi$ we have

$$\psi(f(t) * u) = f(t) * \psi(u)$$

and hence $\psi : M \longrightarrow M$ is an $R[t]$-endomorphism of the left $R[t]$-module ${}_{R[t]}M$. On the other hand, if $\psi : M \longrightarrow M$ is an $R[t]$-endomorphism of ${}_{R[t]}M$, then

$$\psi(\varphi(u)) = \psi(t * u) = t * \psi(u) = \varphi(\psi(u))$$

implies that $\psi \circ \varphi = \varphi \circ \psi$. The centralizer

$$C_\varphi = \{\psi \mid \psi \in \mathrm{Hom}_R(M, M) \text{ and } \psi \circ \varphi = \varphi \circ \psi\}$$

of φ is a $Z(R)$-subalgebra of $\mathrm{Hom}_R(M, M)$ and the argument above gives that

$$C_\varphi = \mathrm{Hom}_{R[t]}(M, M).$$

For a set $\Gamma \neq \varnothing$, the Γ-copower $\coprod_{\gamma\in\Gamma} R[t]$ of the ring $R[t]$ is an ideal of the Γ-direct power ring $\prod_{\gamma\in\Gamma} R[t]$ consisting of all elements $\mathbf{f} = (f_\gamma(t))_{\gamma\in\Gamma}$ with a finite set $\{\gamma \in \Gamma \mid f_\gamma(t) \neq 0\}$ of non-zero coordinates. The power (copower) has a natural $(R[t], R[t])$-bimodule structure. If Γ is finite, then

$$(R[t])^\Gamma = \coprod_{\gamma\in\Gamma} R[t] = \prod_{\gamma\in\Gamma} R[t].$$

If $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ is a nilpotent Jordan normal base of ${}_RM$ with respect to a nilpotent endomorphism φ, then for an element $\mathbf{f} = (f_\gamma(t))_{\gamma\in\Gamma}$ with

$$f_\gamma(t) = a_{\gamma,1} + a_{\gamma,2}t + \cdots + a_{\gamma,n_\gamma+1}t^{n_\gamma}$$

the formula

$$\Phi(\mathbf{f}) = \sum_{\gamma\in\Gamma, 1\leq i\leq k_\gamma} a_{\gamma,i}x_{\gamma,i} = \sum_{\gamma\in\Gamma} f_\gamma(t) * x_{\gamma,1}$$

defines a function

$$\Phi : \coprod_{\gamma\in\Gamma} R[t] \longrightarrow M.$$

In Section 4 we shall make use of

$$\Phi(\mathbf{f}) = \sum_{\gamma\in\Gamma} f_\gamma^{(k_\gamma)}(t) * x_{\gamma,1} = \Phi(\mathbf{f}^{(c)}),$$

where $f_\gamma^{(k_\gamma)}(t)$ is the k_γ-cut of $f_\gamma(t)$ and $\mathbf{f}^{(c)} = (f_\gamma^{(k_\gamma)}(t))_{\gamma\in\Gamma}$ (also an element of the copower) is the cut of $\mathbf{f}$ with respect to the given nilpotent Jordan normal base.

3.1.Theorem. *For a nilpotent endomorphism $\varphi \in \mathrm{Hom}_R(M, M)$ the function Φ is a surjective left $R[t]$-homomorphism. We have $\varphi(\Phi(\mathbf{f})) = \Phi(t\mathbf{f})$ for all $\mathbf{f} \in \coprod_{\gamma\in\Gamma} R[t]$ and the kernel*

$$\coprod_{\gamma\in\Gamma} J(R)[t] + (t^{k_\gamma}) \subseteq \ker(\Phi) \lhd_l \prod_{\gamma\in\Gamma} R[t]$$

is a left ideal of the power (and hence of the copower) ring.

Proof. Clearly,

$$\sum_{\gamma\in\Gamma, 1\leq i\leq k_\gamma} Rx_{\gamma,i} = M$$

implies that Φ is surjective. The second part of the defining formula gives that Φ is a left $R[t]$-homomorphism:

$$\Phi(g(t)\mathbf{f}) = \sum_{\gamma\in\Gamma}(g(t)f_\gamma(t)) * x_{\gamma,1} = \sum_{\gamma\in\Gamma}g(t) * (f_\gamma(t) * x_{\gamma,1}) = g(t) * \Phi(\mathbf{f}),$$

where $g(t) \in R[t]$. We also have

$$\varphi(\Phi(\mathbf{f})) = \sum_{\gamma\in\Gamma}\varphi(f_\gamma(t) * x_{\gamma,1}) = \sum_{\gamma\in\Gamma}(tf_\gamma(t)) * x_{\gamma,1} = \Phi(t\mathbf{f}).$$

If $\mathbf{f} \in \coprod\limits_{\gamma\in\Gamma} J(R)[t] + (t^{k_\gamma})$, then

$$f_\gamma(t) = (a_{\gamma,1} + a_{\gamma,2}t + \cdots + a_{\gamma,k_\gamma}t^{k_\gamma-1}) + (a_{\gamma,k_\gamma+1}t^{k_\gamma} + \cdots + a_{\gamma,n_\gamma+1}t^{n_\gamma})$$

with $a_{\gamma,i} \in J(R)$, $1 \le i \le k_\gamma$. Since $Rx_{\gamma,i}$ is simple, we have $J(R)x_{\gamma,i} = \{0\}$. Thus $\varphi^{k_\gamma}(x_{\gamma,1}) = 0$ implies that $f_\gamma(t) * x_{\gamma,1} = 0$, whence $\Phi(\mathbf{f}) = 0$ follows. Take an element $\mathbf{g} = (g_\gamma(t))_{\gamma\in\Gamma}$ of the direct power and suppose that $\Phi(\mathbf{f}) = 0$ in M. Then

$$\bigoplus_{\gamma\in\Gamma, 1\le i\le k_\gamma} Rx_{\gamma,i} = M$$

implies that $a_{\gamma,i}x_{\gamma,i} = 0$ for all $\gamma \in \Gamma$ and $1 \le i \le k_\gamma$. Thus $f_\gamma(t) * x_{\gamma,1} = 0$ for all $\gamma \in \Gamma$. It follows that

$$\Phi(\mathbf{gf}) = \sum_{\gamma\in\Gamma}(g_\gamma(t)f_\gamma(t)) * x_{\gamma,1} = \sum_{\gamma\in\Gamma}g_\gamma(t) * (f_\gamma(t) * x_{\gamma,1}) = 0,$$

whence $\mathbf{gf} \in \ker(\Phi)$ can be deduced.□

If R is a local ring ($R/J(R)$ is a division ring) and $a_{\gamma,i}x_{\gamma,i} = 0$ for some $1 \le i \le k_\gamma$, then $a_{\gamma,i} \in J(R)$. Thus $\Phi(\mathbf{f}) = 0$ implies that

$$f_\gamma(t) = (a_{\gamma,1}+a_{\gamma,2}t+\cdots+a_{\gamma,k_\gamma}t^{k_\gamma-1})+(a_{\gamma,k_\gamma+1}t^{k_\gamma}+\cdots+a_{\gamma,n_\gamma+1}t^{n_\gamma}) \in J(R)[t]+(t^{k_\gamma}).$$

It follows that for local rings we have

$$\ker(\Phi) = \coprod_{\gamma\in\Gamma} J(R)[t] + (t^{k_\gamma}).$$

4. THE CENTRALIZER OF A NILPOTENT ENDOMORPHISM

Let $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \le i \le k_\gamma\}$ be a nilpotent Jordan normal base of ${}_RM$ with respect to the nilpotent endomorphism $\varphi \in \mathrm{Hom}_R(M, M)$. We keep the notations of the previous section and in the rest of the paper we assume that ${}_RM$ is finitely generated, i.e. that Γ is finite.
A linear order on Γ (say $\Gamma = \{1, 2, \ldots, m\}$) allows us to view an element $\mathbf{f} = (f_\gamma(t))_{\gamma\in\Gamma}$ of $(R[t])^\Gamma$ as a $1 \times \Gamma$ matrix (a row vector) over $R[t]$. For a $\Gamma \times \Gamma$ matrix $\mathbf{P} = [p_{\delta,\gamma}(t)]$ in $M_{\Gamma\times\Gamma}(R[t])$ the matrix product

$$\mathbf{fP} = \sum_{\delta\in\Gamma} f_\delta(t)\mathbf{p}_\delta,$$

of $\mathbf{f}$ and $\mathbf{P}$ is a $1 \times \Gamma$ matrix (row vector) in $(R[t])^\Gamma$, where $\mathbf{p}_\delta = (p_{\delta,\gamma}(t))_{\gamma\in\Gamma}$ is the δ-th row vector of $\mathbf{P}$ and

$$(\mathbf{fP})_\gamma = \sum_{\delta\in\Gamma} f_\delta(t)p_{\delta,\gamma}(t).$$

Using the homomorphism $\Phi : \coprod_{\gamma\in\Gamma} R[t] \longrightarrow M$ introduced in Section 3, we define the subset

$$\mathcal{M}(\Phi) = \{\mathbf{P} \in M_{\Gamma\times\Gamma}(R[t]) \mid \mathbf{fP} \in \ker(\Phi) \text{ for all } \mathbf{f} \in \ker(\Phi)\}.$$

If R is a local ring, then we can determine $\mathcal{M}(\Phi)$. Let $\mathbf{e}_\delta \in \ker\Phi$ be the vector with t^{k_δ} in its δ-coordinate and zeros in all other places. If $\mathbf{P} \in \mathcal{M}(\Phi)$, then $\mathbf{e}_\delta\mathbf{P} \in \ker\Phi$ implies that $t^{k_\delta}p_{\delta,\gamma}(t) \in J(R)[t] + (t^{k_\gamma})$. Thus for local rings we have

$$\mathcal{M}(\Phi)=\{\mathbf{P}\in M_{\Gamma\times\Gamma}(R[t])\,|\,\mathbf{P}=[p_{\delta,\gamma}(t)] \text{ and } t^{k_\delta}p_{\delta,\gamma}(t)\in J(R)[t]+(t^{k_\gamma}) \text{ for all } \delta,\gamma\in\Gamma\}$$

and $\mathbf{E}_{\delta,\gamma} \in \mathcal{M}(\Phi)$ for all $\delta,\gamma \in \Gamma$ with $k_\delta \geq k_\gamma$ (where $\mathbf{E}_{\delta,\gamma}$ denotes the $\Gamma \times \Gamma$ standard matrix unit over $R[t]$ with 1 in the (δ,γ) entry and zeros in the other entries).

4.1.Lemma. *$\mathcal{M}(\Phi)$ is a $Z(R)$-subalgebra of $M_{\Gamma\times\Gamma}(R[t])$. For $\mathbf{P} \in \mathcal{M}(\Phi)$ and $\mathbf{f} = (f_\gamma(t))_{\gamma\in\Gamma}$ in $(R[t])^\Gamma$ the formula*

$$\psi_{\mathbf{P}}(\Phi(\mathbf{f})) = \Phi(\mathbf{fP})$$

properly defines an R-endomorphism $\psi_{\mathbf{P}} : M \longrightarrow M$ of ${}_RM$ such that $\psi_{\mathbf{P}} \circ \varphi = \varphi \circ \psi_{\mathbf{P}}$. The assignment $\mathbf{P} \longmapsto \psi_{\mathbf{P}}$ is a

$$\mathcal{M}(\Phi)^{\mathrm{op}} \longrightarrow C_\varphi$$

homomorphism of $Z(R)$-algebras.

Proof. For $\mathbf{P},\mathbf{Q} \in \mathcal{M}(\Phi)$, $\mathbf{f} \in \ker(\Phi)$ and $c \in Z(R)$ we have

$$\Phi(\mathbf{f}(c\mathbf{P})) = \Phi(c(\mathbf{fP})) = c\Phi(\mathbf{fP}) = 0,$$

$$\Phi(\mathbf{f}(\mathbf{P}+\mathbf{Q})) = \Phi(\mathbf{fP}+\mathbf{fQ}) = \Phi(\mathbf{fP}) + \Phi(\mathbf{fQ}) = 0$$

and $\mathbf{fP} \in \ker(\Phi)$ implies that

$$\Phi(\mathbf{f}(\mathbf{PQ})) = \Phi((\mathbf{fP})\mathbf{Q}) = 0,$$

whence $\mathbf{f}(\mathbf{PQ}) \in \ker(\Phi)$ follows. Thus $c\mathbf{P}, \mathbf{P}+\mathbf{Q}, \mathbf{PQ} \in \mathcal{M}(\Phi)$.
Let $\mathbf{g} \in (R[t])^\Gamma$. If $\Phi(\mathbf{f}) = \Phi(\mathbf{g})$, then $\mathbf{f}-\mathbf{g} \in \ker(\Phi)$ implies that $(\mathbf{f}-\mathbf{g})\mathbf{P} \in \ker(\Phi)$, whence $\Phi(\mathbf{fP}) = \Phi(\mathbf{gP})$ follows. Since Φ is surjective, it follows that $\psi_{\mathbf{P}}$ is well defined. It is straightforward to check that

$$\psi_{\mathbf{P}}(\Phi(\mathbf{f}) + \Phi(\mathbf{g})) = \psi_{\mathbf{P}}(\Phi(\mathbf{f})) + \psi_{\mathbf{P}}(\Phi(\mathbf{g})) \text{ and } \psi_{\mathbf{P}}(r\Phi(\mathbf{f})) = r\psi_{\mathbf{P}}(\Phi(\mathbf{f}))$$

for all $\mathbf{f},\mathbf{g} \in (R[t])^\Gamma$ and $r \in R$. Thus $\psi_{\mathbf{P}}$ is an R-endomorphism. In view of

$$\psi_{\mathbf{P}}(\varphi(\Phi(\mathbf{f})) = \psi_{\mathbf{P}}(t * \Phi(\mathbf{f})) = \psi_{\mathbf{P}}(\Phi(t\mathbf{f})) = \Phi((t\mathbf{f})\mathbf{P}) =$$
$$= \Phi(t(\mathbf{fP})) = t * \Phi(\mathbf{fP}) = t * \psi_{\mathbf{P}}(\Phi(\mathbf{f})) = \varphi(\psi_{\mathbf{P}}(\Phi(\mathbf{f}))),$$

the surjectivity of Φ gives that $\psi_{\mathbf{P}} \circ \varphi = \varphi \circ \psi_{\mathbf{P}}$. Clearly,

$$\psi_{c\mathbf{P}} = c\psi_{\mathbf{P}}\ ,\ \psi_{\mathbf{P}+\mathbf{Q}} = \psi_{\mathbf{P}} + \psi_{\mathbf{Q}} \text{ and } \psi_{\mathbf{PQ}} = \psi_{\mathbf{Q}} \circ \psi_{\mathbf{P}}$$

ensure that $\mathbf{P} \longmapsto \psi_{\mathbf{P}}$ is a homomorphism of $Z(R)$-algebras. We deal only with the last identity:

$$\psi_{\mathbf{PQ}}(\Phi(\mathbf{f})) = \Phi(\mathbf{f}(\mathbf{PQ})) = \Phi((\mathbf{fP})\mathbf{Q}) = \psi_{\mathbf{Q}}(\Phi(\mathbf{fP})) = \psi_{\mathbf{Q}}(\psi_{\mathbf{P}}(\Phi(\mathbf{f}))$$

proves our claim.□

4.2.Lemma. *The R-submodule*

$$\mathcal{V}(k_\gamma,\gamma \in \Gamma)=\{\mathbf{P}\in M_{\Gamma\times\Gamma}(R[t])\,|\,\mathbf{P}=[p_{\delta,\gamma}(t)] \text{ and } \deg(p_{\delta,\gamma}(t))\leq k_\gamma -1 \text{ for all } \delta,\gamma\in\Gamma\}$$

of $M_{\Gamma\times\Gamma}(R[t])$ is R-generated by the set $\{t^i\mathbf{E}_{\delta,\gamma} \mid \delta,\gamma\in\Gamma \text{ and } 0\leq i\leq k_\gamma-1\}$ of matrices:

$$\mathcal{V}(k_\gamma,\gamma\in\Gamma)=\sum_{\delta,\gamma\in\Gamma,0\leq i\leq k_\gamma-1}Rt^i\mathbf{E}_{\delta,\gamma}.$$

Proof. If $\mathbf{P}=[p_{\delta,\gamma}(t)]$ and $\mathbf{P}\in\mathcal{V}(k_\gamma,\gamma\in\Gamma)$, then

$$\mathbf{P}=\sum_{\delta,\gamma\in\Gamma}p_{\delta,\gamma}(t)\mathbf{E}_{\delta,\gamma}=\sum_{\delta,\gamma\in\Gamma,0\leq i\leq k_\gamma-1}b_{\delta,\gamma,i}t^i\mathbf{E}_{\delta,\gamma},$$

where $\deg(p_{\delta,\gamma}(t))\leq k_\gamma-1$ and

$$p_{\delta,\gamma}(t)=b_{\delta,\gamma,1}+b_{\delta,\gamma,2}t+\cdots+b_{\delta,\gamma,k_\gamma}t^{k_\gamma-1}.\square$$

4.3.Lemma. *If $\psi\circ\varphi=\varphi\circ\psi$ holds for an R-endomorphism $\psi:M\longrightarrow M$ of ${}_RM$, then there exists a $\Gamma\times\Gamma$ matrix $\mathbf{P}\in\mathcal{M}(\Phi)\cap\mathcal{V}(k_\gamma,\gamma\in\Gamma)$ such that*

$$\psi(\Phi(\mathbf{f}))=\Phi(\mathbf{fP})$$

for all $\mathbf{f}=(f_\gamma(t))_{\gamma\in\Gamma}$ in $(R[t])^\Gamma$.

Proof. Since $\Phi:(R[t])^\Gamma\longrightarrow M$ is surjective, for each $\delta\in\Gamma$ we can find an element $\mathbf{p}_\delta=(p_{\delta,\gamma}(t))_{\gamma\in\Gamma}$ in $(R[t])^\Gamma$ such that $\Phi(\mathbf{p}_\delta)=\psi(x_{\delta,1})$. We have $\Phi(\mathbf{p}_\delta^{(c)})=\psi(x_{\delta,1})$, where $\mathbf{p}_\delta^{(c)}=(p_{\delta,\gamma}^{(k_\gamma)}(t))_{\gamma\in\Gamma}$ is the cut of $\mathbf{p}_\delta$ with respect to the given nilpotent Jordan normal base. The $\Gamma\times\Gamma$ matrix $\mathbf{P}=[p_{\delta,\gamma}^{(k_\gamma)}(t)]$ consisting of the row vectors $\mathbf{p}_\delta^{(c)}$, $\delta\in\Gamma$, is in $\mathcal{V}(k_\gamma,\gamma\in\Gamma)$ and

$$\psi(\Phi(\mathbf{f}))=\sum_{\delta\in\Gamma}\psi(f_\delta(t)*x_{\delta,1})=\sum_{\delta\in\Gamma}f_\delta(t)*\psi(x_{\delta,1})=\sum_{\delta\in\Gamma}f_\delta(t)*\Phi(\mathbf{p}_\delta^{(c)})=$$
$$=\sum_{\delta\in\Gamma}\Phi(f_\delta(t)\mathbf{p}_\delta^{(c)})=\Phi(\sum_{\delta\in\Gamma}f_\delta(t)\mathbf{p}_\delta^{(c)})=\Phi(\mathbf{fP})$$

for all $\mathbf{f}\in(R[t])^\Gamma$. Since $\mathbf{f}\in\ker(\Phi)$ implies that $\Phi(\mathbf{fP})=\psi(\Phi(\mathbf{f}))=0$, we obtain that $\mathbf{P}\in\mathcal{M}(\Phi)$.$\square$

4.4.Theorem. *Let $\varphi:M\longrightarrow M$ be a nilpotent R-endomorphism of the finitely generated semisimple left R-module ${}_RM$. Then the centralizer C_φ (as a $Z(R)$-subalgebra of $\mathrm{Hom}_R(M,M)$) is the homomorphic image of the opposite of some $Z(R)$-subalgebra of the matrix algebra $M_{m\times m}(R[t])$, where $m=\dim_R(\ker(\varphi))$.*

Proof. Lemma 4.1 ensures that $\mathbf{P}\longmapsto\psi_{\mathbf{P}}$ is a

$$\mathcal{M}(\Phi)^{\mathrm{op}}\longrightarrow C_\varphi$$

homomorphism of $Z(R)$-algebras. Since our assignment is surjective by Lemma 4.3, we obtain that C_φ is the homomorphic image of $\mathcal{M}(\Phi)^{\mathrm{op}}$, where $\mathcal{M}(\Phi)$ is a $Z(R)$-subalgebra of $M_{\Gamma\times\Gamma}(R[t])$. To conclude the proof it is enough to note that $m=\dim_R(\ker(\varphi))=|\Gamma|$.$\square$

4.5.Corollary. *Let $\varphi:M\longrightarrow M$ be a nilpotent R-endomorphism of the finitely generated semisimple left R-module ${}_RM$. Then C_φ satisfies all of the polynomial identities (with coefficients in $Z(R)$) of $M_{m\times m}^{\mathrm{op}}(R[t])$. If R is commutative, then C_φ satisfies the standard identity $S_{2m}=0$ of degree $2m$ by the Amitsur-Levitzki theorem.*

4.6.Theorem. *Let $\varphi : M \longrightarrow M$ be a nilpotent R-endomorphism of the finitely generated semisimple left R-module ${}_RM$. Then the centralizer C_φ (as a $Z(R)$-submodule of $\mathrm{Hom}_R(M,M)$) is the homomorphic image of some $Z(R)$-submodule of a certain md-generated R-submodule of $M_{m\times m}(R[t])$, where $d = \dim_R(M)$ and $m = \dim_R(\ker(\varphi))$.*

Proof. Lemma 4.1 and 4.3 ensure that $\mathbf{P} \longmapsto \psi_{\mathbf{P}}$ is a surjective

$$\mathcal{M}(\Phi) \cap \mathcal{V}(k_\gamma, \gamma \in \Gamma) \longrightarrow C_\varphi$$

homomorphism of $Z(R)$-modules, where $\mathcal{M}(\Phi)\cap\mathcal{V}(k_\gamma, \gamma \in \Gamma)$ is a $Z(R)$-submodule of the left R-submodule $\mathcal{V}(k_\gamma, \gamma \in \Gamma)$ of $M_{\Gamma\times\Gamma}(R[t])$. Using Lemma 4.2, we obtain that $\mathcal{V}(k_\gamma, \gamma \in \Gamma)$ is R-generated by the set $\{t^i\mathbf{E}_{\delta,\gamma} \mid \delta, \gamma \in \Gamma \text{ and } 0 \le i \le k_\gamma - 1\}$ having

$$|\Gamma| \cdot \sum_{\gamma\in\Gamma} k_\gamma = md$$

elements.□

4.7.Corollary. *Let $\varphi : M \longrightarrow M$ be a nilpotent D-linear map of the finite dimensional left vector space ${}_DM$ over a division ring D. If D is finite dimensional over its central subfield $K = Z(D)$, then C_φ is a K-subspace of $\mathrm{Hom}_D(M,M)$ and*

$$\dim_K(C_\varphi) \le \dim_K(D) \cdot \dim_D(\ker(\varphi)) \cdot \dim_D(M).$$

Remark. If $D = K$ is an algebraically closed field and all Jordan blocks of the nilpotent linear map $\varphi \in \mathrm{Hom}_K(M,M)$ (of the finite dimensional K-vector space M) are of the same size $s \times s$, then we can use the formula for $\dim(C_{A_i})$ in Section 1 to see that the upper bound in the above Corollary 4.7 is sharp:

$$\dim_K(C_\varphi) = m^2 s = md = \dim_K(\ker(\varphi)) \cdot \dim_K(M).$$

4.8.Theorem. *If ${}_RM$ is semisimple and $\varphi : M \longrightarrow M$ is an indecomposable nilpotent element of the ring $\mathrm{Hom}_R(M,M)$, then the following are equivalent.*

(1) $\psi \in C_\varphi$.

(2) *We can find an R-generating set $\{y_j \in M \mid 1 \le j \le d\}$ of ${}_RM$ and elements $a_1, a_2, \ldots, a_n$ in R such that*

$$a_1y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \psi(y_j)$$

and

$$a_1\varphi(y_j) + a_2\varphi(\varphi(y_j)) + \cdots + a_n\varphi^{n-1}(\varphi(y_j)) = \psi(\varphi(y_j))$$

for all $1 \le j \le d$.

Proof.
(1)$\Longrightarrow$(2): Obviously, if $\psi \in C_\varphi$ then the first identity implies the second one. Proposition 2.3 ensures the existence of a nilpotent Jordan normal base $\{x_i \mid 1 \le i \le n\}$ of ${}_RM$ with respect to φ consisting of one block. Clearly, $\bigoplus_{1\le i\le n} Rx_i = M$ implies that

$$\psi(x_1) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_1x_1 + a_2\varphi(x_1) + \cdots + a_n\varphi^{n-1}(x_1)$$

for some $a_1, a_2, \ldots, a_n \in R$. Thus

$$\psi(x_i) = \psi(\varphi^{i-1}(x_1)) = \varphi^{i-1}(\psi(x_1)) = \varphi^{i-1}(a_1x_1 + a_2\varphi(x_1) + \cdots + a_n\varphi^{n-1}(x_1)) =$$

$=a_1\varphi^{i-1}(x_1)+a_2\varphi(\varphi^{i-1}(x_1))+\cdots+a_n\varphi^{n-1}(\varphi^{i-1}(x_1))=a_1x_i+a_2\varphi(x_i)+\cdots+a_n\varphi^{n-1}(x_i)$

for all $1\leq i\leq n$.

(2)$\Longrightarrow$(1): Since we have

$$\varphi(\psi(y_j))=\varphi(a_1y_j+a_2\varphi(y_j)+\cdots+a_n\varphi^{n-1}(y_j))=$$

$$=a_1\varphi(y_j)+a_2\varphi(\varphi(y_j))+\cdots+a_n\varphi^{n-1}(\varphi(y_j))=\psi(\varphi(y_j))$$

for all $1\leq j\leq d$, the implication is proved.$\square$

4.9.Corollary. *If R is commutative, ${}_RM$ is semisimple and $\varphi:M\longrightarrow M$ is an indecomposable nilpotent element of the ring $\mathrm{Hom}_R(M,M)$, then the following are equivalent.*

1. *$\psi\in C_\varphi$.*
2. *We can find elements $a_1,a_2,\ldots,a_n$ in R such that*

$$a_1u+a_2\varphi(u)+\cdots+a_n\varphi^{n-1}(u)=\psi(u)$$

for all $u\in M$. In other words ψ is a polynomial of φ.

Proof. It suffices to prove that if $\sum\limits_{1\leq j\leq d}Ry_j=M$ and

$$a_1y_j+a_2\varphi(y_j)+\cdots+a_n\varphi^{n-1}(y_j)=\psi(y_j)$$

holds for all $1\leq j\leq d$, then we have

$$a_1u+a_2\varphi(u)+\cdots+a_n\varphi^{n-1}(u)=\psi(u)$$

for all $u\in M$. Since $u=b_1y_1+b_2y_2+\cdots+b_dy_d$ for some $b_1,b_2,\ldots,b_d\in R$ and $b_ja_i=a_ib_j$, we obtain that

$$\psi(u)=\sum_{1\leq j\leq d}b_j\psi(y_j)=\sum_{1\leq j\leq d}b_j(a_1y_j+a_2\varphi(y_j)+\cdots+a_n\varphi^{n-1}(y_j))=$$

$$=a_1\left(\sum_{1\leq j\leq d}b_jy_j\right)+a_2\varphi\left(\sum_{1\leq j\leq d}b_jy_j\right)+\cdots+a_n\varphi^{n-1}\left(\sum_{1\leq j\leq d}b_jy_j\right)=$$

$$=a_1u+a_2\varphi(u)+\cdots+a_n\varphi^{n-1}(u).\square$$

4.10.Theorem. *Let R be a local ring. If $\varphi:M\longrightarrow M$ is a nilpotent R-endomorphism of the finitely generated semisimple left R-module ${}_RM$ and $\sigma\in\mathrm{Hom}_R(M,M)$ is arbitrary, then the following are equivalent.*

1. *$C_\varphi\subseteq C_\sigma$.*
2. *We can find an R-generating set $\{y_j\in M\mid 1\leq j\leq d\}$ of ${}_RM$ and elements $a_1,a_2,\ldots,a_n$ in R such that*

$$a_1\psi(y_j)+a_2\varphi(\psi(y_j))+\cdots+a_n\varphi^{n-1}(\psi(y_j))=\sigma(\psi(y_j))$$

for all $1\leq j\leq d$ and all $\psi\in C_\varphi$.

Proof.
(1)$\Longrightarrow$(2): Obviously, if $C_\varphi \subseteq C_\sigma$ then

$$a_1 y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \sigma(y_j)$$

implies that

$$a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$$

for all $\psi \in C_\varphi$. Theorem 2.1 ensures the existence of a nilpotent Jordan normal base $\{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ of ${}_RM$ with respect to φ. Consider the natural projection $\varepsilon_\delta : M \longrightarrow N'_\delta$ corresponding to the direct sum $M = N'_\delta \oplus N''_\delta$ (see the proof of 2.3), where

$$N'_\delta = \bigoplus_{1\leq i\leq k_\delta} Rx_{\delta,i} \text{ and } N''_\delta = \bigoplus_{\gamma\in\Gamma\setminus\{\delta\},1\leq i\leq k_\gamma} Rx_{\gamma,i}\ .$$

Then $\varepsilon_\delta \in C_\varphi$, whence $\varepsilon_\delta \in C_\sigma$ follows for all $\delta \in \Gamma$. Thus $\operatorname{im}(\varepsilon_\delta) = N'_\delta$ and

$$\sigma : \operatorname{im}(\varepsilon_\delta) \longrightarrow \operatorname{im}(\varepsilon_\delta)$$

implies that

$$\sigma(x_{\delta,1}) = \sum_{1\leq i\leq k_\delta} a_{\delta,i}x_{\delta,i} = h_\delta(t) * x_{\delta,1}$$

for some $h_\delta(t) = a_{\delta,1} + a_{\delta,2}t + \cdots + a_{\delta,k_\delta}t^{k_\delta-1}$ in $R[t]$. Since $\varphi \in C_\sigma$ implies that $\sigma \in C_\varphi$, it follows that

$$\sigma(\Phi(\mathbf{f})) = \sum_{\gamma\in\Gamma}\sigma(f_\gamma(t) * x_{\gamma,1}) = \sum_{\gamma\in\Gamma}f_\gamma(t) * \sigma(x_{\gamma,1}) = \sum_{\gamma\in\Gamma}f_\gamma(t) * (h_\gamma(t) * x_{\gamma,1}) =$$

$$= \sum_{\gamma\in\Gamma}(f_\gamma(t)h_\gamma(t)) * x_{\gamma,1} = \Phi(\mathbf{fH}),$$

where $\mathbf{f} \in (R[t])^\Gamma$ and $\mathbf{H} = \sum\limits_{\gamma\in\Gamma} h_\gamma(t)\mathbf{E}_{\gamma,\gamma}$ is a $\Gamma\times\Gamma$ diagonal matrix in $\mathcal{M}(\Phi)$ (note that $\mathbf{H} \in\mathcal{M}(\Phi)$ is a consequence of $\sigma(\Phi(\mathbf{f})) = \Phi(\mathbf{fH})$). In view of Lemma 4.1 and 4.3, the containment $C_\varphi \subseteq C_\sigma$ is equivalent to the condition that $\sigma \circ \psi_{\mathbf{P}} = \psi_{\mathbf{P}} \circ \sigma$ for all $\mathbf{P} \in \mathcal{M}(\Phi)$. Consequently, we obtain that $C_\varphi \subseteq C_\sigma$ is equivalent to the following:

$$\Phi(\mathbf{fPH}) = \sigma(\Phi(\mathbf{fP}))) = \sigma(\psi_{\mathbf{P}}(\Phi(\mathbf{f}))) = \psi_{\mathbf{P}}(\sigma(\Phi(\mathbf{f}))) = \psi_{\mathbf{P}}(\Phi(\mathbf{fH})) = \Phi(\mathbf{fHP})$$

for all $\mathbf{f} \in (R[t])^\Gamma$ and $\mathbf{P} \in \mathcal{M}(\Phi)$. Thus $C_\varphi \subseteq C_\sigma$ implies that $\Phi(\mathbf{f}(\mathbf{PH}-\mathbf{HP})) = 0$, i.e. that $\mathbf{f}(\mathbf{PH} - \mathbf{HP}) \in \ker(\Phi)$ for all $\mathbf{f}$ and $\mathbf{P}$.
Now we use that R is a local ring. If $\alpha \in \Gamma$ is an index such that

$$k_\alpha = \max\{k_\gamma \mid \gamma \in \Gamma\} = n,$$

then $\mathbf{E}_{\alpha,\delta} \in \mathcal{M}(\Phi)$ for all $\delta \in \Gamma$ (see the argument preceding Lemma 4.1). Take $\mathbf{e} = (1)_{\gamma\in\Gamma}$ and $\mathbf{P} = \mathbf{E}_{\alpha,\delta}$, then the δ-coordinate of

$$\mathbf{e}(\mathbf{E}_{\alpha,\delta}\mathbf{H} - \mathbf{H}\mathbf{E}_{\alpha,\delta}) = (h_\delta(t) - h_\alpha(t))\mathbf{e}\mathbf{E}_{\alpha,\delta}$$

is $h_\delta(t) - h_\alpha(t)$. Since

$$(h_\delta(t) - h_\alpha(t))\mathbf{e}\mathbf{E}_{\alpha,\delta} \in \ker(\Phi) = \coprod_{\gamma\in\Gamma} J(R)[t] + (t^{k_\gamma}),$$

we obtain that $h_\delta(t) - h_\alpha(t) \in J(R)[t] + (t^{k_\delta})$. Thus

$$\sigma(x_{\delta,1}) = h_\delta(t) * x_{\delta,1} = h_\alpha(t) * x_{\delta,1}$$

for all $\delta \in \Gamma$. It follows that

$$\sigma(x_{\gamma,i}) = \sigma(\varphi^{i-1}(x_{\gamma,1})) = \varphi^{i-1}(\sigma(x_{\gamma,1})) = \varphi^{i-1}(h_\alpha(t) * x_{\gamma,1}) =$$
$$= h_\alpha(t) * \varphi^{i-1}(x_{\gamma,1}) = h_\alpha(t) * x_{\gamma,i} = a_1 x_{\gamma,i} + a_2\varphi(x_{\gamma,i}) + \cdots + a_n\varphi^{n-1}(x_{\gamma,i}),$$

where $h_\alpha(t) = a_1 + a_2 t + \cdots + a_n t^{n-1}$.
(2)$\Longrightarrow$(1): (R is an arbitrary ring) Since $1_M \in C_\varphi$, we obtain that

$$a_1 y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \sigma(y_j)$$

for all $1 \le j \le d$. If $\psi \in C_\varphi$, then

$$\psi(\sigma(y_j)) = \psi(a_1 y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j)) =$$
$$= a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$$

for all $1 \le j \le d$, whence $\psi \circ \sigma = \sigma \circ \psi$ follows. Thus $C_\varphi \subseteq C_\sigma$.$\square$

5. THE CENTRALIZER OF AN ARBITRARY LINEAR MAP

Let $r \ge 1$ be an integer such that

$$\ker(\varphi^r) \oplus \operatorname{im}(\varphi^r) = M$$

for the R-endomorphism $\varphi \in \mathrm{Hom}_R(M, M)$ of the left R-module ${}_R M$. Since

$$\varphi : \ker(\varphi^r) \longrightarrow \ker(\varphi^r) \text{ and } \varphi : \operatorname{im}(\varphi^r) \longrightarrow \operatorname{im}(\varphi^r),$$

we can take the restricted R-endomorphisms

$$\varphi_1 = \varphi \upharpoonright \ker(\varphi^r) \text{ and } \varphi_2 = \varphi \upharpoonright \operatorname{im}(\varphi^r).$$

In the next statement we consider the following centralizers

$C_\varphi \subseteq \mathrm{Hom}_R(M, M), C_{\varphi_1} \subseteq \mathrm{Hom}_R(\ker(\varphi^r), \ker(\varphi^r)), C_{\varphi_2} \subseteq \mathrm{Hom}_R(\operatorname{im}(\varphi^r), \operatorname{im}(\varphi^r))$.

5.1.Lemma. *If* $\ker(\varphi^r) \oplus \operatorname{im}(\varphi^r) = M$ *holds for* $\varphi \in \mathrm{Hom}_R(M, M)$, *then we have an isomorphism*

$$C_\varphi \cong C_{\varphi_1} \times C_{\varphi_2}$$

of $Z(R)$*-algebras.*

Proof. Consider the natural projections

$$\varepsilon_1 : M \longrightarrow \ker(\varphi^r) \text{ and } \varepsilon_2 : M \longrightarrow \operatorname{im}(\varphi^r)$$

and the natural injections

$$\tau_1 : \ker(\varphi^r) \longrightarrow M \text{ and } \tau_2 : \operatorname{im}(\varphi^r) \longrightarrow M.$$

For $\sigma \in C_\varphi$ define an assignment by

$$\sigma \longmapsto (\varepsilon_1\sigma\tau_1, \varepsilon_2\sigma\tau_2)$$

and for a pair $(\sigma_1, \sigma_2) \in C_{\varphi_1} \times C_{\varphi_2}$ define an assignment by

$$(\sigma_1, \sigma_2) \longmapsto \tau_1\sigma_1\varepsilon_1 + \tau_2\sigma_2\varepsilon_2.$$

Since $\sigma \in C_\varphi$ ensures that

$$\sigma(\ker(\varphi^r)) \subseteq \ker(\varphi^r) \text{ and } \sigma(\operatorname{im}(\varphi^r)) \subseteq \operatorname{im}(\varphi^r),$$

it is straightforward to see that the above definitions provide

$$C_\varphi \longrightarrow C_{\varphi_1} \times C_{\varphi_2} \text{ and } C_{\varphi_1} \times C_{\varphi_2} \longrightarrow C_\varphi$$

homomorphisms of $Z(R)$-algebras which are mutual inverses of each other.$\square$

5.2.Theorem. *Let K be an algebraically closed field and $\varphi : V \longrightarrow V$ be a K-linear map of the finite dimensional vector space V. If*

$$\{\lambda_1, \lambda_2, \dots, \lambda_p\} \subseteq K$$

is the (p-element) set of all eigenvalues of φ and

$$m_i = \dim(\ker(\varphi - \lambda_i 1_V)),$$

then for each $1 \leq i \leq p$ there exists a K-subalgebra $\mathcal{M}_i$ of the full $m_i \times m_i$ matrix algebra $M_{m_i \times m_i}(K[t])$ such that the centralizer C_φ is the homomorphic image of the direct product K-algebra

$$\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_p.$$

Proof. We apply induction on the dimension of V.
If $\dim V = 1$, then $V = Kx$ and $\mathrm{Hom}_K(V, V) \cong K$ (as K-algebras) is commutative. Now $\varphi = \lambda 1_V$ and $\dim(\ker(\varphi - \lambda 1_V)) = 1$, where λ is the (only) eigenvalue of φ. Thus $C_\varphi =\mathrm{Hom}_K(V, V) \cong M_{1\times1}(K)$ and $M_{1\times1}(K)$ is a K-subalgebra of $M_{1\times1}(K[t])$.
Let $n \geq 2$ be an integer and assume that our theorem holds for all vector spaces over K of dimension less or equal than $n - 1$. Consider the situation described in the theorem with $\dim V = n$. Since K is algebraically closed, we have a (non empty) p-element set

$$\{\lambda_1, \lambda_2, \dots, \lambda_p\} \subseteq K$$

of eigenvalues of φ. Let $0 \neq u \in V$ be an eigenvector with $\varphi(u) = \lambda_1 u$. The Fitting lemma ensures the existence of an integer $r \geq 1$ such that

$$\ker(\varphi - \lambda_1 1_V)^r \oplus \mathrm{im}(\varphi - \lambda_1 1_V)^r = V.$$

Clearly,

$$\varphi - \lambda_1 1_V : \ker(\varphi - \lambda_1 1_V)^r \longrightarrow \ker(\varphi - \lambda_1 1_V)^r,$$

thus the restricted function

$$\varphi_1 = (\varphi - \lambda_1 1_V) \restriction \ker(\varphi - \lambda_1 1_V)^r$$

is a nilpotent K-linear map of the K-subspace $\ker(\varphi - \lambda_1 1_V)^r \leq V$. The application of Theorem 4.4 provides a K-subalgebra $\mathcal{N}_1$ of the matrix algebra $M_{m_1\times m_1}(K[t])$ with $m_1 = \dim(\ker(\varphi_1))$ such that the centralizer

$$C_{\varphi_1} \subseteq \mathrm{Hom}_K(\ker(\varphi - \lambda_1 1_V)^r, \ker(\varphi - \lambda_1 1_V)^r)$$

is the K-homomorphic image of the opposite K-algebra $\mathcal{N}_1^{\mathrm{op}}$. Using

$$\ker(\varphi - \lambda_1 1_V) \subseteq \ker(\varphi - \lambda_1 1_V)^r,$$

we obtain that $\ker(\varphi_1) = \ker(\varphi - \lambda_1 1_V)$.
Since $(\varphi - \lambda_1 1_V)(u) = 0$ implies that $u \in \ker(\varphi - \lambda_1 1_V)^r$, we obtain that $\mathrm{im}(\varphi - \lambda_1 1_V)^r$ is a proper K-subspace of V, i.e. that $\mathrm{im}(\varphi - \lambda_1 1_V)^r \neq V$ and hence $\dim(\mathrm{im}(\varphi - \lambda_1 1_V)^r) \leq n - 1$.
In view of

$$\varphi - \lambda_1 1_V : \mathrm{im}(\varphi - \lambda_1 1_V)^r \longrightarrow \mathrm{im}(\varphi - \lambda_1 1_V)^r,$$

the application of the induction hypothesis on the restricted K-linear map

$$\varphi_2 = (\varphi - \lambda_1 1_V) \restriction \mathrm{im}(\varphi - \lambda_1 1_V)^r$$

yields the existence of K-subalgebras $\mathcal{N}_j \leq M_{m'_j\times m'_j}(K[t])$, $2 \leq j \leq q$, with

$$m'_j = \dim(\ker(\varphi_2 - \mu_j 1_{\mathrm{im}(\varphi - \lambda_1 1_V)^r}))$$

such that the centralizer

$$C_{\varphi_2} \subseteq \mathrm{Hom}_K(\mathrm{im}(\varphi - \lambda_1 1_V)^r, \mathrm{im}(\varphi - \lambda_1 1_V)^r)$$

is the homomorphic image of the direct product K-algebra

$$\mathcal{N}_2 \times \mathcal{N}_3 \times \cdots \times \mathcal{N}_q.$$

Note that

$$\{\mu_2, \mu_3, \ldots, \mu_q\} \subseteq K$$

is the set of all eigenvalues of φ_2. Now

$$\ker(\varphi_2 - \mu_j 1_{\mathrm{im}(\varphi-\lambda_1 1_V)^r}) = \ker((\varphi - \lambda_1 1_V) - \mu_j 1_V) \cap \mathrm{im}(\varphi - \lambda_1 1_V)^r \subseteq \ker((\varphi - (\lambda_1 + \mu_j) 1_V)$$

implies that

$$m'_j = \dim(\ker(\varphi_2 - \mu_j 1_{\mathrm{im}(\varphi-\lambda_1 1_V)^r})) \leq \dim(\ker((\varphi - (\lambda_1 + \mu_j) 1_V)) = m_{i(j)},$$

where $\lambda_1 + \mu_j = \lambda_{i(j)}$ for some unique $2 \leq i(j) \leq p$. Indeed, each $\lambda_1 + \mu_j$, $2 \leq j \leq q$, is an eigenvalue of φ and $\mu_j = 0$ would imply that $(\varphi - \lambda_1 1_V)(v) = \varphi_2(v) = 0$ for some $0 \neq v \in \mathrm{im}(\varphi - \lambda_1 1_V)^r$ in contradiction with

$$\ker(\varphi - \lambda_1 1_V) \cap \mathrm{im}(\varphi - \lambda_1 1_V)^r \subseteq \ker(\varphi - \lambda_1 1_V)^r \cap \mathrm{im}(\varphi - \lambda_1 1_V)^r = \{0\}.$$

Thus $C_{\varphi_1} \times C_{\varphi_2}$ is the homomorphic image of the direct product K-algebra

$$\mathcal{N}_1^{\mathrm{op}} \times \mathcal{N}_2 \times \cdots \times \mathcal{N}_q$$

and $m'_j \leq m_{i(j)}$ allows us to view the K-subalgebra $\mathcal{N}_j$ of $M_{m'_j \times m'_j}(K[t])$ as a K-subalgebra of $M_{m_{i(j)} \times m_{i(j)}}(K[t])$. Using Proposition 5.1, we deduce that the centralizer $C_{\varphi - \lambda_1 1_V} = C_\varphi$ is the homomorphic image of the following direct product of K-algebras

$$\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_p,$$

where $\mathcal{M}_1 = \mathcal{N}_1^{\mathrm{op}}$, $\mathcal{M}_{i(j)} = \mathcal{N}_j$ for $2 \leq j \leq q$ and $\mathcal{M}_i = \{0\}$ if

$$i \in \{1, 2, \ldots, p\} \setminus \{i(2), i(3), \ldots, i(q)\}.$$

Since $K[t]$ is commutative, the transpose involution ensures that $M^{\mathrm{op}}_{m_1 \times m_1}(K[t]) \cong M_{m_1 \times m_1}(K[t])$ as K-algebras. Thus the opposite K-algebra $\mathcal{N}_1^{\mathrm{op}}$ also can be considered as a K-subalgebra of $M_{m_1 \times m_1}(K[t])$ consisting of the transposed matrices: $\mathcal{N}_1^{\mathrm{op}} \cong \mathcal{N}_1^{\top}$.□

5.3.Corollary. *Let K be an algebraically closed field and $\varphi : V \longrightarrow V$ be a K-linear map of the finite dimensional vector space V. If*

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subseteq K$$

is the (p-element) set of all eigenvalues of φ and

$$m = \max\{\dim(\ker(\varphi - \lambda_i 1_V)) \mid 1 \leq i \leq p\}$$

then the centralizer C_φ satisfies the polynomial identities of the full $m \times m$ matrix algebra $M_{m\times m}(K[t])$.

Proof. Since $m_i = \dim(\ker(\varphi - \lambda_i 1_V)) \leq m$, the $m_i \times m_i$ matrix algebra $M_{m_i \times m_i}(K[t])$ satisfies the polynomial identities of the $m \times m$ matrix algebra $M_{m\times m}(K[t])$. Thus each $\mathcal{M}_i$ satisfies the polynomial identities of the $m \times m$ matrix algebra $M_{m\times m}(K[t])$, whence we obtain that the same holds for any homomorphic image of their direct product.□

Remark. If A is an $n \times n$ matrix over an algebraically closed field K with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$, then $\dim(\ker(A - \lambda_i I))$ is the number of blocks in the Jordan normal form of A containing λ_i in the diagonal. Since $\mathrm{Hom}_K(V, V) \cong M_{n\times n}(K)$, Corollary 5.3 says that from a PI point of view the centralizer $C_A \subseteq M_{n\times n}(K)$ of A behaves like a matrix ring (over the polynomial ring $K[t]$) of size much smaller than n. If the characteristic polynomial of A coincides with the minimal polynomial, then $m_i = \dim(\ker(A-\lambda_i I)) = 1$ for each $1 \leq i \leq p$. Thus $m = \max\{m_i \mid 1 \leq i \leq p\} = 1$ and Corollary 5.3 gives the commutativity of C_A. For such A we have

$$C_A = \{f(A) \mid f(t) \in K[t]\},$$

whence the commutativity of C_A also follows (more details in Section 1).

REFERENCES

(1) Bergman, G.: *Centralizers in free associative algebras*, Trans. Amer. Math. Soc. 137 (1969), 327-344.
(2) Guralnick, R.: *A note on commuting pairs of matrices*, Linear and Multilinear Algebra 31 (1992), 71-75.
(3) Guralnick, R., Sethuraman, B.A.: *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl. 310 (2000), 139–148.
(4) Nelson, G.C., Ton-That, T.: *Multiplicatively closed bases for C(A)*, Note di Matematica 26, n. 2 (2006), 81–104.
(5) Prasolov, V.V.: *Problems and Theorems in Linear Algebra*, Vol. 134 of Translation of Mathematical Monographs, American Mathematical Society, Providence, Rhode Island, 1994.
(6) Suprunenko, D.A. and Tyshkevich, R.I.: *Commutative Matrices*, Academic Press, New York and London, 1968.
(7) Szigeti, J.: *Linear algebra in lattices and nilpotent endomorphisms of semisimple modules*, Journal of Algebra 319 (2008), 296–308.

Institute of Mathematics, University of Miskolc, Miskolc, Hungary 3515
E-mail address: jeno.szigeti@uni-miskolc.hu

Department of Mathematical Sciences, Stellenbosch University, P/Bag X1, Matieland 7602, South Africa
E-mail address: lvw@sun.ac.za